Strategies for Integration

Please note that instead of "u-substitution" I refer to "w-substitution". This is to avoid possible confusion if you end up needing to use the integration by parts formula.

- Always begin by checking whether the problem can be done by w-substitution. If this does not work, the ways in which it fails can help you pick the next best strategy.

  1. If you have a parentheses to an exponent (other than ± 1), let $w =$ the inside and see if the power rule applies.
  2. If the parenthesis has an exponent of -1 or if there is a denominator whose exponent is 1, let $w =$ the inside of the parentheses or the denominator (same thing), and see if $\int \frac{1}{w} dw = \ln |w| + C$ applies.
  3. If you have an e (or anything of the form a^u where a is a constant and u is a function of x):
     - if there is only one e, try letting w be the exponent on e.
     - if there is more than one e, see if one of the other strategies applies.
  4. If your problem has a trig function and NONE of the other strategies applies, let $w =$ the angle. If this does not work, see if there are any identities that you can use to rewrite the problem in a better form.

- If the integrand has a $u^2 + a^2$, $u^2 - a^2$, or a $a^2 - u^2$ in it (where u is a function of x and a is a constant), possibly raised to some power, use trig substitution. You want to label your triangle in such a way that $\frac{u}{a}$ equals one of the following: $\tan\theta$, $\sec\theta$, or $\sin\theta$. The following accomplish this, listed in the same order.

  - If you have a $u^2 + a^2$, label the hypotenuse $\sqrt{u^2 + a^2}$, the leg opposite θ is u and the leg adjacent is a.
  - If you have a $u^2 - a^2$, label the hypotenuse u, the leg opposite θ is $\sqrt{u^2 - a^2}$ and the leg adjacent is a.
  - If you have a $a^2 - u^2$, label the hypotenuse a, the leg opposite θ is u and the leg adjacent is $\sqrt{a^2 - u^2}$.

- If the integrand is a quotient of two polynomials, $\frac{f(x)}{g(x)}$, use partial fractions to simplify. If the degree of the numerator is $\geq$ the degree of the denominator, do long division first. This sometimes even reduces the problem to just a w-substitution problem! :)

  1. Factor the denominator as much as is possible.
  2. For linear factors that occur only once, $u_1 \ldots u_n$, write $\frac{A_1}{u_1} + \cdots + \frac{A_n}{u_n}$.
  3. For any linear factors that occur twice, say t_1, use two quotients of the as in $\frac{T_1}{t_1} + \frac{T_2}{t_1^2}$. Add these to any from step one.
  4. For irreducible quadratic factors, say q_1 you need a linear form in the numerator as in $\frac{Q_1 x + Q_2}{q_1}$. Add these to anything you get from steps two and three.

5. Add all of these together by getting a common denominator and gather like x-terms together in the numerator.
6. Set the original numerator equal to the new one.
7. Use the fact that the coefficients of like powers of x on each side must be equal to each other to get equations involving the "new variable" - the unknown constants.
8. Solve the system of equations and plug back into the original equation. Now integrate each part.

- If w-substitution fails due to one of the following, use integration by parts. That is: $\int udv = uv - \int vdu$.
  - The integrand has extra x's not accounted for in the w and dw. In this case put all the extra x's in the u and everything else (i.e. what was in your w and dw) in the dv. Realize that you may have to use integration by parts several times to get rid of all of the x's.
  - The integrand is a function (like ln) that you do not know how to integrate, but you can differentiate. Put the function in u and everything else in dv.
  - The integrand is a product of two functions, neither of which reduce in complexity when differentiated (like $e^{f(x)}$ and trig functions). Then let one function be u and the other dv. Apply integration by parts (usually twice) and solve for the original integral.
  - The integrand is an inverse trig function, $trig^{-1}x$. You need to do a change of variables by letting $w = trig^{-1}(x)$, so that $trig(w) = x$. Now find dx and substitute. Once the change of variable is complete, you should be in the first scenario.
- If the integral is a definite proper integral, follow the above steps to integrate and then evaluate accordingly.
- If the integral is a definite integral involving an infinity either as a limit of integration or as a vertical asymptote (VA) in the range of the limits of integration, then it is an improper integral. Express the integral as a limit as follows and then follow the above strategies to integrate.
  - $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$.
  - $\int_{-\infty}^b f(x)dx = \lim_{a\to\infty} \int_a^b f(x)dx$.
  - $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^b f(x)dx + \int_a^\infty f(x)dx$ and then use the above.
  - If you have $\int_a^b f(x)dx$ with
    * the VA at a then $\lim_{c\to a^+} \int_c^b f(x)dx$.
    * the VA at b then $\lim_{c\to b^-} \int_a^c f(x)dx$.
    * the VA at c, $a < c < b$ then use $\int_a^c f(x)dx + \int_c^b f(x)dx$ with the above.