Homework Three

Due at the beginning of class, Monday, February 23.

- (1.7.16) Let G be a group and let $A = G$. Show that the maps defined by $g \cdot a = gag^{-1}$ for all $g, a \in G$ satisfy the axioms of a (left) group action of G on itself (this action of G on itself is called *conjugation*).

- (1.7.17) Let G be a group and let G act on itself by left conjugation, so each $g \in G$ maps G to G by
$$x \mapsto gxg^{-1}.$$
For fixed $g \in G$, prove that conjugation by g is an isomorphism from G onto itself (i.e. is an automorphism of G).

- (1.7.18) Let H be a group acting on a set A. Prove that the relation $\sim$ on A defined by
$$a \sim b \qquad \text{if and only if} \qquad a = hb \text{ for some } h \in H$$
is an equivalence relation. (For each $x \in A$ the equivalence class of x under $\sim$ is called the *orbit* of x under the action of H. The orbits under the action of H partition the set A.)

- (1.7.19) Let H be a subgroup of a finite group G and let H act on G (here $A = G$) by left multiplication. Let $x \in G$ and let $\mathcal{O}$ be the orbit of x under the action of H. Prove that the map
$$H \to \mathcal{O} \qquad \text{defined by} \qquad h \mapsto hx$$
is a bijection (hence all orbits have cardinality $|H|$). From this and the preceeding exercise deduce *Lagrange's Theorem*:

  if G is a finite group and H is a subgroup of G then $|H|$ divides $|G|$.