Algebra Midterm Exam Two

1 Definitions and Quick Questions

Choose five out of the first six problems.

1. Define normality for a subgroup N of a group G, and list three equivalent conditions.

   A subgroup N of G is normal if $gNg^{-1} = N$ for all $g \in G$. Equivalently:

   - $gng^{-1} \in N$ for all $g \in G$ and $n \in N$.
   - $gNhN = ghN$ for all $g, h \in G$.
   - $gN = Ng$ for all $g \in G$.
   - N is the kernel of a group homomorphism $f : G \to H$ for some group H.
   - $N_G(N) = G$.

2. Let G be a group and $H \leq G$. Given a fixed element $g \in G$, what is the left coset gH?

$$gH = \{gh | g \in G, h \in H\}$$

3. Let G be a group and $H \leq G$. What is $[G : H]$?

   $[G : H]$ is the number of distinct left cosets of H in G.

4. Let $G = D_8$ and $H =< r^2, s >$. Compute gH for each $g \in D_8$.

Here $H = \{1, r^2, s, sr^2\}$. The cosets are:

1	$\{1, r^2, s, sr^2\} = H$
r	$\{r, r^3, sr^3, sr\} = D_8 \setminus H$
r^2	$\{1, r^2, s, sr^2\} = H$
r^3	$\{r, r^3, sr^3, sr\} = D_8 \setminus H$
s	$\{1, r^2, s, sr^2\} = H$
sr	$\{r, r^3, sr^3, sr\} = D_8 \setminus H$
sr^2	$\{1, r^2, s, sr^2\} = H$
sr^3	$\{r, r^3, sr^3, sr\} = D_8 \setminus H$

5. Consider the group $\mathbb{Z}/12\mathbb{Z}$. Fill in the table with the orders of each of the elements:

Recall that for $G =< g >$ cyclic of order n, we have $|g^k| = \frac{n}{gcd(n,k)}$

Element	Order
0	1
1	12
2	6
3	4
4	3
5	12
6	2
7	12
8	3
9	4
10	6
11	12

6. State Lagrange's theorem.

Let G be a group and $H \leq G$. Then $|G| = |H|[G : H]$.

2 Proofs

7. Let A and B be subgroups of G such that $A \leq B \leq G$. Prove that

$$[G : A] = [G : B][B : A].$$

By Lagrange's theorem we have $|G| = |B|[G : B]$. But B itself is a group, with subgroup A, so again by Lagrange we have $|B| = |A|[B : A]$. Substituting we have

$$|G| = |B|[G : B] = |A|[B : A][G : B].$$

However, also by Lagrange we have:

$$|G| = |A|[G : A]$$

and thus we can equate $|A|[B : A][G : B]$ and $|A|[G : A]$. From this it immediately follows (divide by $|A|$ and rearrange terms) that $[G : A] = [G : B][B : A]$.

8. If G has no normal subgroups other than G and $\{1_G\}$, show that every $f : G \to H$ is either trivial (i.e. $f(g) = 1_H \, \forall g \in G$) or one-to-one.

Let $f : G \to H$ be a group homomorphism. We know that $ker(f)$ is a normal subgroup of G. By hypothesis, there are only two of these, G and $\{1_G\}$. If $ker(f) = G$ we have $f(g) = 1_H$ for all $g \in G$, and hence f is trivial. On the other hand, if we have $ker(f) = \{1_G\}$ then f is one-to-one.

9. If $H < G$ is of order n, and H is the only subgoup of G of order n, prove that $H \triangleleft G$.

We have seen that gHg^{-1} is itself a subgroup of G. (For completeness, if you like, let gh_1g^{-1} and gh_2g^{-1} be in gHg^{-1}. Note that $(gh_2g^{-1})^{-1} = gh_2^{-1}g^{-1}$, and we have $gh_1g^{-1}gh_2^{-1}g^{-1} = gh_1h_2^{-1}g^{-1} \in gHg^{-1}$, so gHg^{-1} is indeed a subgroup of G. You didn't need to show this, as we already did it on the board.) Note also that $|gHg^{-1}| = |H| = n$, since if $gh_1g^{-1} = gh_2g^{-1}$ then $h_1 = h_2$ by cancellation. Since H is the only subgroup of G of order n, it must be the case then that $gHg^{-1} = H$, whence $H \triangleleft G$.

10. Suppose that A and B are two normal subgroups of G and that $A \cap B = \{1_G\}$. Show that $ab = ba$ for any $a \in A$, $b \in B$,

If $A \triangleleft G$ and $B \triangleleft G$ we note that $aba^{-1} \in B$ for any choice of b and that $ba^{-1}b^{-1} \in A$ for any choice of $a^{-1} \in A$. Notice then that

$$(aba^{-1})b^{-1} = b'b^{-1} \in B$$

and

$$a(ba^{-1}b^{-1}) = aa' \in A$$

which implies that $aba^{-1}b^{-1} \in A \cap B$. By assumption $A \cap B = \{1_G\}$, so $aba^{-1}b^{-1} = 1_G$. Multiplying both sides on the right by ba we have $ab = ba$.