Theorem 1. *Let $f : A \to B$ be a function. There is a set A/f and there are functions $g : A \to A/f$ and $h : A/f \to B$ such that $f = g \circ h$, g is onto, and h is one-to-one.*

Proof. Define $\sim$ on A by $a_1 \sim a_2$ if $f(a_1) = f(a_2)$. Clearly $\sim$ is reflexive and symmetric. If $a_1 \sim a_2$ and $a_2 \sim a_3$ then $f(a_1) = f(a_2) = f(a_3)$, so $a_1 \sim a_3$ and $\sim$ is transitive, and hence an equivalence relation. Define A/f to be the set of all equivalence classes under $\sim$, and define $g : A \to A/f$ by $g(a) = [a] = \{x \in A | f(x) = f(a)\}$. To see that g is onto, choose $[x] \in A/f$. By definition $g(x) = [x]$, so g is onto.

Now let $[a] \in A/f$, $[a] = \{x \in A | f(x) = f(a)\}$. Define $h([a]) = f(a)$. To see that h is well defined, suppose $x, y \in [a]$. Then $f(x) = f(a)$ and $f(y) = f(a)$, so $h([a]) = f(x) = f(y) = f(a)$, and h does not depend on the choice of equivalence class member.

It remains to show that h is one-to-one and that $f = h \circ g$. To see that h is one-to-one, suppose that $h([a_1]) = h([a_2]) = b$. Then $h([a_1]) = f(a_1) = b = h([a_2]) = f(a_2)$, so $f(a_1) = f(a_2)$ and $a_1 \sim a_2$, whence $[a_1] = [a_2]$, and h is one-to-one. Finally let $a \in A$. Then $g(a) = [a]$ and $h([a]) = f(a)$, so $h(g(a)) = f(a)$ and $f = h \circ g$. ☐