Let A, B, and C be subsets of a universal set U.

1. $A \subseteq A \cup B$

   Proof. We want to show that if $x \in A$ then $x \in A \cup B$. Suppose that $x \in A$. By definition of union, $x \in A \cup B$ if $x \in A$ or $x \in B$. If $x \in A$ is true then $x \in A$ or $x \in B$ must be true as well, so $x \in A \cup B$. Thus $A \subseteq A \cup B$ ☐

2. $A \cap B \subseteq A$

   Proof. Let $x \in A \cap B$. By definition of intersection $x \in A$ and $x \in B$. It is clear then that $x \in A$, so we've established that if $x \in A \cap B$ then $x \in A$ and hence $A \cap B \subseteq A$. ☐

3. $A \setminus B \subseteq A$

   Proof. Let $x \in A \setminus B$. Then $x \in A$ and $x \notin B$ by definition. In particular $x \in A$, so we have that if $x \in A \setminus B$ then $x \in A$, and hence $A \setminus B \subseteq A$. ☐

4. $(A \setminus B) \cap (B \setminus A) = \emptyset$

   Proof. Suppose for the sake of contradiction that the intersection is not empty. Then there is an $x \in (A \setminus B) \cap (B \setminus A)$. Thus $x \in A \setminus B$ and $x \in B \setminus A$. By definition of set difference, from $x \in A \setminus B$ we infer that $x \in A$ and $x \notin B$. Also by definition of set difference, from $x \in B \setminus A$ we infer that $x \in B$ and $x \notin A$. At this point we have been forced to conclude that $x \in A$ and $x \notin A$, which is clearly a contradiction. Our assumption then, that the intersection was non-empty must by incorrect, and hece we've established by contradiction that $A \setminus B$ and $B \setminus A$ are disjoint. ☐

5. A and $\overline{A}$ are disjoint.

   Proof. By contradiction. Assume that there is some $x \in A \cap \overline{A}$. Then we have $x \in A$ and $x \in \overline{A}$. By definition of complement, $x \in \overline{A}$ implies $x \notin A$, and we now have a contradiction: $x \in A$ and $x \notin A$. Our assumption must have been false, and we are forced to conclude that $A \cap \overline{A} = \emptyset$. ☐

6. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

   Proof. Suppose that $A \subseteq B$ and $B \subseteq C$. We want to show that $A \subseteq C$ and so need to establish that if $x \in A$ then $x \in C$. Suppose, then, that $x \in A$. Since $A \subseteq B$ we can infer that $x \in B$. But since $B \subseteq C$, $x \in B$ implies $x \in C$. Thus we have established that if $x \in A$ then $x \in C$, and hence $A \subseteq C$. ☐

7. $A \cup B = B \cup A$ and $A \cap B = B \cap A$

   Proof. First we show that $A \cup B \subseteq B \cup A$. Let $x \in A \cup B$. Then either $x \in A$ or $x \in B$. If $x \in A$ then $x \in B \cup A$, and on the other hand if $x \in B$ then $x \in B \cup A$. In either case then $x \in B \cup A$ and we have $A \cup B \subseteq B \cup A$.

Next we let $y \in B \cup A$. Then either $y \in B$ or $y \in A$. In the first case, if $y \in B$ then $y \in A \cup B$ and in the second case if $y \in A$ then $y \in A \cup B$. We have thus established that if $y \in B \cup A$ then $y \in A \cup B$, and hence $B \cup A \subseteq A \cup B$.

Now since $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$ we have $A \cup B = B \cup A$. $\square$

Proof. Begin by assuming that $x \in A \cap B$. By definition of intersection we have $x \in A$ and $x \in B$. Since $x \in B$ and $x \in A$ we have $x \in B \cap A$, and thus $A \cap B \subseteq B \cap A$.

Conversely, suppose that $y \in B \cap A$. By definition of intersection we have $y \in B$ and $y \in A$. Since $y \in A$ and $y \in B$ we know that $y \in A \cap B$, and thus we have $B \cap A \subseteq A \cap B$.

Now since $A \cap B \subseteq B \cap A$ and $B \cap A \subseteq A \cap B$ we have $A \cap B = B \cap A$. $\square$

8. $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$

Proof. First consider the set A. By (1), $A \subseteq (A \cup B)$, and also by (1) $(A \cup B) \subseteq (A \cup B) \cup C$. Thus by (6) $A \subseteq (A \cup B) \cup C$. Also by (1) $A \subseteq A \cup (B \cup C)$.

Next considering the set B we have, by (1), $B \subseteq (B \cup A)$, and by (7) $(B \cup A) = (A \cup B)$, so $B \subseteq (A \cup B)$. Again by (1) $(A \cup B) \subseteq (A \cup B) \cup C$ so we have, by (6), $B \subseteq (A \cup B) \cup C$. Also by (1) we know $B \subseteq (B \cup C)$, and again by (1) $(B \cup C) \subseteq (B \cup C) \cup A$, which by (7) equals $A \cup (B \cup C)$. Thus we also have $B \subseteq A \cup (B \cup C)$.

Finally considering the set C note that by (1) and (7) we have $C \subseteq (A \cup B) \cup C$. Again by (1) and (7) we have $C \subseteq (B \cup C)$, which, by (1) and (7), is contained in $A \cup (B \cup C)$, and hence by (6) $C \subseteq A \cup (B \cup C)$.

Note that $x \in A \cup (B \cup C)$ implies that $x \in A$ or $x \in (B \cup C)$, which means $x \in B$ or $x \in C$. Thus if $x \in A \cup (B \cup C)$ then $x \in A$ or $x \in B$ or $x \in C$. In each case above we've established that all three are subsets of $(A \cup B) \cup C$, so in any case $x \in (A \cup B) \cup C$. From this we infer that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Similarly if $x \in (A \cup B) \cup C$ then $x \in (A \cup B)$ or $x \in C$. If $x \in (A \cup B)$ then $x \in A$ or $x \in B$, and again we see that if $x \in (A \cup B) \cup C$ then $x \in A$ or $x \in B$ or $x \in C$. As above, we showed that each of these is a subset of $A \cup (B \cup C)$, so in any case $x \in A \cup (B \cup C)$. From this we infer that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$, and we have established that $A \cup (B \cup C) = (A \cup B) \cup C$. $\square$

Proof. We begin by noting that $X \cap Y \subseteq Y$ by (7) and (2). Thus we have $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.

Now $A \cap (B \cap C) \subseteq A$ and $A \cap (B \cap C) \subseteq (B \cap C) \subseteq B$. Thus $A \cap (B \cap C) \subseteq (A \cap B)$. Also $A \cap (B \cap C) \subseteq (B \cap C) \subseteq C$, so $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

Conversely $(A \cap B) \cap C \subseteq (A \cap B) \subseteq A$. Also $(A \cap B) \cap C \subseteq (A \cap B) \subseteq B$ and $(A \cap B) \cap C \subseteq C$ so $(A \cap B) \cap C \subseteq (B \cap C)$. Thus $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

We have now established that $A \cap (B \cap C) = (A \cap B) \cap C$ since each is a subset of the other. $\square$

9. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   Done in class.

10. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

    Done in class (several different ways!).

11. $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$

    Proof. Suppose first that $A \subseteq B$. We want to show that $\overline{B} \subseteq \overline{A}$. Let $x \in \overline{B}$. We will show that $x \in \overline{A}$ by contradiction, so suppose instead that $x \notin \overline{A}$, which means, of course that $x \in A$. Since we have assumed that $A \subseteq B$, $x \in A$ implies $x \in B$. We have assumed, however that $x \in \overline{B}$, so $x \notin B$, and we've arrived at a contradiction, and are forced to conclude that $x \notin \overline{A}$ cannot be the case, whence $x \in \overline{A}$. At this point we've established that $A \subseteq B$ implies $\overline{B} \subseteq \overline{A}$.

    On the other hand, suppose that $\overline{B} \subseteq \overline{A}$. We want to show that $A \subseteq B$. Suppose, for the sake of contradiction, that $A \not\subseteq B$. Then there must be some $a \in A$ such that $a \notin B$. Then since $a \notin B$ we have $a \in \overline{B}$. By hypothesis, $\overline{B} \subseteq \overline{A}$, so $a \in \overline{A}$, implying $a \notin A$. But we begin with $a \in A$, and thus have a contradiction, implying that our assumption that $A \not\subseteq B$ must be false, and we are forced to conclude that $A \subseteq B$. ☐

    Proof. Suppose that $A \subseteq B$. By definition this means that if $x \in A$ then $x \in B$. This statement is equivalent to its contrapositive, that is, if $x \notin B$ then $x \notin A$. By definition of complement, this is the same as if $x \in \overline{B}$ then $x \in \overline{A}$, which is the definition of $\overline{B} \subseteq \overline{A}$. ☐

12. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

13. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

14. $\overline{\overline{A}} = A$ $((A^c)^c = A)$

    Proof. Let $x \in \overline{\overline{A}}$. By definition of complement, $x \notin \overline{A}$. By definition $y \in \overline{A}$ if and only if $y \notin A$. We note then that $x \notin \overline{A}$ means that $x \in A$, since if $x \notin A$ then $x \in \overline{A}$. So if $x \in \overline{\overline{A}}$ then $x \in A$, and thus $\overline{\overline{A}} \subseteq A$.

    Conversely let $x \in A$. We claim that this means that $x \notin \overline{A}$, for if $x \in \overline{A}$ then $x \notin A$, a contradiction. Thus we see that $x \in A$ implies $x \notin \overline{A}$, and hence by definition of complement, $x \in \overline{\overline{A}}$, and hence $A \subseteq \overline{\overline{A}}$. ☐

15. $A \cup \emptyset = A$

    Proof. By (1), $A \subseteq A \cup \emptyset$. Now let $x \in A \cup \emptyset$. In a union, two cases arise, $x \in A$ or $x \in \emptyset$, the second of which is clearly impossible here. Thus we must conclude that $x \in A$, and hence $A \cup \emptyset \subseteq A$. Having proved containment in both directions, we now have $A \cup \emptyset = A$. ☐

16. $A \cap \emptyset = \emptyset$

    Proof. By (2), $A \cap \emptyset \subseteq \emptyset$. We note that the statement "If $x \in \emptyset$ then $x \in S$" is vacuously true for any set S (the antecedent is always true), and hence the empy set is always a subset of any set, including $A \cap \emptyset$. Thus $A \cap \emptyset = A$.

17. $U \cup A = U$

    Proof. By definition of U, $U \cup A \subseteq U$. By (1), $U \subseteq U \cup A$. Thus $U \cup A = U$. □

18. $U \cap A = A$

    Proof. By (2), $U \cap A \subseteq A$. Let $x \in A$. By definition of U, $x \in U$, so we have $x \in A$ and $x \in U$, whence $x \in U \cap A$. Thus $A \subseteq U \cap A$. □

Note that many of these could be proved more easily if we used the following lemmas that we discussed in class at some point, but not before we started this exercise:

Lemma 1. *Let* $A \subseteq B$ *and* $A \subseteq C$. *Then* $A \subseteq (B \cap C)$.

Proof. Suppose that $A \subseteq B$ and $A \subseteq C$. Let $x \in A$. By definition of subset, since $A \subseteq B$ we know that $x \in B$. Similarly, since $A \subseteq C$ we know that $x \in C$. Thus $x \in B$ and $x \in C$, whence $x \in B \cap C$, and we have $A \subseteq B \cap C$. □

Lemma 2. *Let* $A \subseteq C$ *and* $B \subseteq C$. *Then* $A \cup B \subseteq C$.

Proof. Suppose that $A \subseteq C$ and $B \subseteq C$. Let $x \in A \cup B$. Two cases arise, either $x \in A$ or $x \in B$. If $x \in A$ then since $A \subseteq C$ we know that $x \in C$. On the other hand if $x \in B$ then since $B \subseteq C$ we know that $x \in C$. In either case we have $x \in C$, so $A \cup B \subseteq C$. □