Homework Two

1. Prove that the product of any two consecutive natural numbers, plus the larger of the two numbers, results in a perfect square.

   Proof. Let n and $n+1$ denote the consecutive natural numbers. Clearly $n+1$ is the larger of the two. Now

   $$\begin{aligned} n \cdot (n+1) + n + 1 &= n^2 + n + n + 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

   which is obviously a perfect square, and we have established the result. □

Definition 1. *A natural number n is* even *if $2|n$. If n is not even we say that n is* odd*.*

2. Prove that a natural number n is even if and only if $n+1$ is odd.

   Proof. Let n be even. Then $n = 2k$ for some natural number k. Note then that $n+1 = 2k+1$, which is not divisible by 2, and hence is odd.

   Conversely let $n+1$ be odd. Then $n+1 = 2\ell + 1$ for some ℓ. Subtracting one we obtain $n = 2\ell$, which is even. □

3. Prove directly that if m and n are odd then mn is odd.

   Proof. Let m and n be odd. Then $m = 2k+1$ and $n = 2\ell + 1$ for some natural numbers k and ℓ. Multiplying we have

   $$\begin{aligned} mn &= (2k+1)(2\ell+1) \\ &= 4k\ell + 2k + 2\ell + 1 \qquad = 2(2k\ell + k + \ell) + 1 \end{aligned}$$

   which is odd. □

4. Prove by contradiction that if m and n are odd then mn is odd.

   Proof. Suppose that $m = 2k+1$ and $n = 2\ell+1$ are odd, but mn is even. Then there must be some intger x such that $mn = 2x$. Then $mn = (2k+1)(2\ell+1) = 2(2k\ell + k + \ell) + 1 = 2x$. Subtracting we obtain

   $$\begin{aligned} 0 &= 2(2k\ell + k + \ell) + 1 - 2k \\ &= 2(2k\ell + k + \ell - k) + 1 \\ &= 2(2k\ell + \ell) + 1 \\ -1 &= 2(2k\ell + \ell) \end{aligned}$$

   implying that $2|(-1)$, a contradiction. □

Proof. (Shorter, but really this is a proof by contrapositive hidden in proof by contradiction's clothing.) Suppose that $m = 2k + 1$ and $n = 2\ell + 1$ are odd, but mn is even. Then there must be some intger x such that $mn = 2x$. Then $mn = (2k + 1)(2\ell + 1) = 2(2k\ell + k + \ell) + 1$, which is odd, and we now have mn both even and odd, a contradiction. □

5. Prove that if every even natural number greater than 2 is the sum of two primes then every odd natural number greater than 5 is the sum of three primes.

   Proof. Assume that every even natural number greater than 2 can be expressed as the sum of two primes. Let n be odd and greater than 5. We note that $n-3$ is thus greater than 2, and if n is odd, say $n = 2k + 1$ then $n - 3 = 2k + 1 - 3 = 2k - 2 = 2(k - 1)$ is even. Therefore by hypothesis $n - 3 = p + q$ with p and q prime, and hence $n = n - 3 + 3 = p + q + 3$ is the sum of three primes. □

Your illustrious professor has decided to undertake a serious scientific study of "free time" for the remainder of the semester. In order to free up some time to do this, he has left the chore of grading (certainly one of a math professor's most thankless undertakings) to you. You are to use the following scale:

- Assign a grade of R (= right) if the claim and proof are correct (even if the proof is not the simplest or most elegant or shortest or the one you would have given).
- Assign a grade of W (= wrong) if the claim is incorrect, or if the main idea in the alleged proof given is incorrect, or if most of the statements contained within the alleged proof are incorrect.
- Assign a grade of B (= in between) for a proof that is largely correct but contains one or two incorrect statements or justifications.

Students, of course, like to have feedback on their work, so whenever you assign a grade of W or B, you need to explain your grade. Tell what is incorrect and why.

6. *Claim* If m^2 is odd then m is odd.

   Proof Assume m is odd. Then $m = 2k + 1$ for some integer k. From this we have that $m^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. Factoring a 4 out of the first two terms we have $m^2 = 4(k^2 + k) + 1 = 2(2(k^2 + k)) + 1$, which is odd. Thus if m^2 is odd then m is odd. □

   W The student proved the converse, which is not logically equivalent to the original statement. (The proof of the converse is correct, however.)

7. *Claim* If $a|b$ and $a|c$ and $b > c$ then $a|(b-c)$.

Proof Assume that $a|b$ and $a|c$ but $a \not| (b-c)$. Then there is no natural number q such that $aq = (b-c)$. Now since $a|b$ there is an m such that $am = b$ and since $a|c$ there is an n such that $an = c$. Thus $b - c = am - an = a(m-n)$. Thus we have that $a|(b-c)$. But we assumed that $a \not| (b-c)$. Here we have a contradiction, and we are forced to conclude that if $a|b$ and $a|c$ and $b > c$ then $a|(b-c)$. □

R The proof by contradiction more or less contains within it a direct proof, but is nonetheless valid.