Homework Four

Prove or give a counterexample:

1. If $A \subseteq B$ and $C \subseteq D$ then $A \cup C \subseteq B \cup D$.

   Answer taken directly from student work:

   Proof. Assume $A \subseteq B$ and $C \subseteq D$. Let $x \in A \cup C$. Then we know either $x \in A$ or $x \in C$. If $x \in A$ then, since $A \subseteq B$, we know $x \in B$, and thus $x \in B \cup D$. Now if $x \in C$ then, since $C \subseteq D$, we know $x \in D$ and thus $x \in B \cup D$. Therefore if $A \subseteq B$ and $C \subseteq D$ then $A \cup C \subseteq B \cup D$. $\square$

2. If $A \subseteq B$ and $C \subseteq D$ then $A \cap C \subseteq B \cap D$.

   Answer taken directly from student work:

   Proof. Assume $A \subseteq B$ and $C \subseteq D$. Then by rule, $A \cap C \subseteq A$, since $A \subseteq B$ then $A \cap C \subseteq A \subseteq B$, and $A \cap C \subseteq B$. Again by rule $A \cap C \subseteq C$, this time we see that $A \cap C \subseteq C \subseteq D$. We now see that $A \cap C \subseteq B$ and $A \cap C \subseteq D$ so $A \cap C \subseteq B \cap D$. $\square$

3. If $A \cap B = A \cap C$ then $B = C$.

   Answer taken directly from student work (answers will vary):

   Counterexample:

$$
\begin{aligned}
A &= \{1, 2, 3\} \\
B &= \{2, 3, 4\} \\
C &= \{2, 3, 5\}
\end{aligned}
$$

   Now $A \cap B = \{2, 3\} = A \cap C$ but $B \neq C$.

Your illustrious professor has decided to undertake a serious scientific study of "free time" for the remainder of the semester. In order to free up some time to do this, he has left the chore of grading (certainly one of a math professor's most thankless undertakings) to you. You are to use the following scale:

- Assign a grade of R (= right) if the claim and proof are correct (even if the proof is not the simplest or most elegant or shortest or the one you would have given).
- Assign a grade of W (= wrong) if the claim is incorrect, or if the main idea in the alleged proof given is incorrect, or if most of the statements contained within the alleged proof are incorrect.

- Assign a grade of B (= in between) for a proof that is largely correct but contains one or two incorrect statements or justifications.

Students, of course, like to have feedback on their work, so whenever you assign a grade of W or B, you need to explain your grade. Tell what is incorrect and why.

4. If $A \cup B = A \cup C$ then $B = C$.

   Proof By contradiction. Suppose that $B \neq C$. Then there is some $b \in B$ such that $b \notin C$. Since $b \in B$ it must be the case that $b \in A \cup B$. But since $b \notin C$ it must also be the case that $b \notin A \cup C$. Therefore $b \in A \cup B$ but $b \notin A \cup C$. Since sets are equal if and only if they have the same elements, it must be the case that $A \cup B \neq A \cup C$, and we have established the contrapositive. $\square$