

Homework 6

Due at the beginning of class on Monday, March 22.

Definition 1. Given sets A and B , and a relation $R \subseteq A \times B$, define the relation $R^{-1} \subseteq B \times A$ by $(b, a) \in R^{-1}$ if $(a, b) \in R$.

Definition 2. Given sets A and B , and a relation $R \subseteq A \times B$, we say that the relation R is onto if for every $b \in B$ there is some $a \in A$ with $(a, b) \in R$.

1. Let $R \subseteq A \times A$ be a partial order relation (i.e. R is reflexive, antisymmetric, and transitive) on a set A . Prove that R^{-1} is also a partial order relation on A .

Proof. Since $(a, a) \in R$ for all $a \in A$, $(a, a) \in R^{-1}$ as well, so R^{-1} is reflexive.

Now let (a, b) and $(b, a) \in R^{-1}$. Then (b, a) and $(a, b) \in R$. Since R is antisymmetric, $a = b$, implying that R^{-1} is antisymmetric.

Finally suppose that (a, b) and (b, c) are in R^{-1} . Then (c, b) and (b, a) are in R , the transitivity of which implies $(c, a) \in R$, giving us $(a, c) \in R^{-1}$. We now have R^{-1} reflexive, antisymmetric, and transitive, so R^{-1} is a partial order relation. \square

2. Prove or give a counterexample to each of the following:

- (a) If $R \subseteq A \times A$ and $S \subseteq A \times A$ are transitive relations then $R \cup S$ is a transitive relation.

Let $A = \{a, b, c, d, e\}$ and let $R = \{(a, b), (b, c), (a, c)\}$ and $S = \{(c, d), (d, e), (c, e)\}$. Then both R and S are transitive, but (a, c) and (c, e) are in $R \cup S$, but (a, e) is not, and hence $R \cup S$ fails to be transitive.

- (b) If $R \subseteq A \times A$ and $S \subseteq A \times A$ are transitive relations then $R \cap S$ is a transitive relation.

Proof. Let (a, b) and (b, c) be in $R \cap S$. Then $(a, b), (b, c) \in R$, the transitivity of which implies $(a, c) \in R$. Further we have (a, b) and $(b, c) \in S$, whose transitivity implies $(a, c) \in S$. At this point we have $(a, c) \in R$ and $(a, c) \in S$, implying $(a, c) \in R \cap S$, and hence $R \cap S$ is transitive. \square

3. If $R \subseteq A \times B$ and $S \subseteq B \times C$ are relations, define the relation $S \circ R \subseteq A \times C$ by $(a, c) \in S \circ R$ if there is $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$.

- (a) Prove that if R and S are both onto then $S \circ R$ is onto.

Proof. Let $c \in C$. Since S is onto there is $b \in B$ with $(b, c) \in S$. Since R is onto there is $a \in A$ with $(a, b) \in R$, and hence $(a, c) \in S \circ R$, which is then onto. \square

- (b) Prove that if $S \circ R$ is onto then S is onto.

Proof. Let $c \in C$. Since $S \circ R$ is onto there is $a \in A$ with $(a, c) \in S \circ R$. By definition this means that there is $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$, so there is $b \in B$ with $(b, c) \in S$, implying that S is onto. \square

- (c) Give an example to show that if $S \circ R$ being onto need not imply that R is onto. Let $A = \{a\}$, $B = \{b_1, b_2\}$, and $C = \{c\}$. Define $R = (a, b_1)$ and $S = (b_1, c)$. Then $S \circ R = (a, c)$, which is onto. Since there is no pair $(-, b_2)$ in R , however, R is not onto. \square