Definition 1. *Let $f : A \to B$ be a function and let $T \subseteq B$. Define $f^{\hookleftarrow}(T) = \{a \in A | f(a) \in f(T)\}$. Note that sometimes $f^{-1}(T)$ is used instead to denote this, but can be confusing as we are not referring to an inverse function.*

1. Let $f : A \to B$ be a function and let $S \subseteq A$. Prove that $S \subseteq f^{\hookleftarrow}(f(S))$ and give an example showing that this containment can be proper (i.e. $S \subset f^{\hookleftarrow}(f(S))$).

   Proof. Let $s \in S$. Then by definition $f(s) \in f(S)$. Since $s \in \{a \in A | f(a) = f(s) \in f(S)\}$ it follows that $s \in f^{\hookleftarrow}(f(S))$, and hence $S \subseteq f^{\hookleftarrow}(f(S))$. □

   To see that this containment can be proper, consider

   $$A = \{1, 2\}, \quad B = \{b\}, \quad , S = \{1\}, \qquad f(1) = f(2) = b.$$

   Now $f(S) = \{b\}$ and $f^{\hookleftarrow}(f(S)) = \{1, 2\}$, of which $S = \{1\}$ is a proper subset.

2. What condition(s) are needed on f to ensure that $S = f^{\hookleftarrow}(f(S))$? State and prove your result.

   If f is one to one then $f^{\hookleftarrow}(f(S)) = S$.

   Proof. One direction of the containment was proved above. For the other direction, let $a \in f^{\hookleftarrow}(f(S))$. Then $f(a) = b$ where $b \in f(S)$, implying that there is some $s \in S$ such that $f(s) = b$. But now we have $f(a) = b$ and $f(s) = b$, and since f is one-to-one, a must equal s, and hence $a \in S$, and therefore $f^{\hookleftarrow}(f(S)) \subseteq S$. □

3. Let f be a function from a set A to a set B, and let S and T be subsets of A.

   (a) If f is onto is it the case that $f(S \cap T) = f(S) \cap f(T)$? Prove this or give a counterexample.

   For a counterexample consider the following:

   $$A = \{1, 2, 3\}, \quad S = \{1, 2\}, \quad T = \{2, 3\}, \qquad B = \{x, y\}, \qquad f(1) = f(3) = x, \; f(2) = y.$$

   Here we have $f(S \cap T) = f(\{2\}) = y$ and $f(S) = f(T) = f(S \cap T) = \{x, y\}$.

   (b) If f is one-to-one is it the case that $f(S \cap T) = f(S) \cap f(T)$? Prove this or give a counterexample.

   Proof. Let $b \in f(S \cap T)$. Then there is $a \in S \cap T$ with $f(a) = b$. Then $a \in S$ and hence $b = f(a) \in f(S)$ and $a \in T$ and hence $b = f(a) \in f(T)$, so $b \in (f(S) \cap f(T))$ and $f(S \cap T) \subseteq (f(S) \cap f(T))$. Note that we did not use one-to-one in this direction.

   For the other containment let $y \in (f(S) \cap f(T))$. Since $y \in f(S)$ there is $s \in S$ such that $f(s) = y$. Also since $y \in f(T)$ there is $t \in T$ such that $f(t) = y$. Now since f is one-to-one and $f(s) = f(t) = y$ it must be the case that $s = t$, and hence $s = t \in S \cap T$ and $y \in f(S \cap T)$. □

4. For each element of S_5, draw a function diagram indicating where each element (1-5) goes:

   (a) (132)(42)(541)(23)

x	$f(x)$
1	5
2	2
3	4
4	3
5	1

   (b) (23)(34)(45)

x	$f(x)$
1	1
2	3
3	4
4	5
5	2

   (c) (14)(13)(12)

x	$f(x)$
1	2
2	3
3	4
4	1
5	5

5. Consider the function f from the open interval $(0,1)$ to the set of real numbers $\mathbb{R}$ defined by the following formula:

$$f(x) = \begin{cases} \frac{x-1/2}{x} & \text{for } x \leq 1/2 \\ \frac{x-1/2}{1-x} & \text{for } x > 1/2. \end{cases}.$$

   Prove that f is one-to-one and onto.

   Proof. We first note that for $x < \frac{1}{2}$, $f(x) < 0$ since $x - 1/2$ will be negative and the denominator (x) will be positive. Of course $f(1/2) = 0$, and for $x > \frac{1}{2}$, $f(x) > 0$ since $x - 1/2$ will be positive and the denominator $(1-x)$ will be positive.

   We will now prove that f is one-to-one. Toward this end, suppose that $a, b \in (0,1)$ with $f(a) = f(b)$. If $f(a) = f(b) = 0$ then $a = b = 1/2$ (this is the only way to make the numerator of either expression zero). If $f(a) = f(b) < 0$ it must be the case that both a and b are between 0 and 1/2. From this we see that:

$$\begin{aligned} f(a) &= f(b) \\ \frac{a-1/2}{a} &= \frac{b-1/2}{b} \\ b(a-1/2) &= a(b-1/2) \\ ab-\frac{b}{2} &= ab-\frac{a}{2} \\ \frac{b}{2} &= \frac{a}{2} \\ b &= a. \end{aligned}$$

Finally if $f(a) = f(b) > 0$ it must be the case that both a and b are between 1/2 and 1. From this we see that:

$$\begin{aligned} f(a) &= f(b) \\ \frac{a-1/2}{1-a} &= \frac{b-1/2}{1-b} \\ (1-b)(a-1/2) &= (1-a)(b-1/2) \\ a-\frac{1}{2}-ab+\frac{b}{2} &= b-\frac{1}{2}-ab+\frac{a}{2} \\ a+\frac{b}{2} &= b+\frac{a}{2} \\ a-\frac{a}{2} &= b-\frac{b}{2} \\ \frac{a}{2} &= \frac{b}{2} \\ a &= b. \end{aligned}$$

We see then that in either case we have $f(a) = f(b)$ implying $a = b$, and hence f is one to one.

To see that f is onto let $y \in \mathbb{R}$. If $y = 0$ then $f(1/2) = y$. If $y < 0$ consider $x = \frac{-1}{2y-2}$. Note that both numerator and denominator are negative here, so $x > 0$ and since $y < 0$, $2y < 0$ so $2y - 2 < -2$, and hence $x < 1/2$. Plugging in then $\frac{x-1/2}{x} = y$. Finally if $y > 0$ then consider $x = \frac{2y+1}{2y+2}$. Now clearly $2y + 1 < 2y + 2$, so $x < 1$, and if we compute $x - 1/2$ we obtain (after some algebra) $\frac{y}{2y+2}$, which, since $y > 0$, is positive, implying $x > 1/2$. Thus we hvae $f(x) = \frac{x-1/2}{1-x} = y$. We are thus able to find, for any $y \in \mathbb{R}$, and $x \in (0, 1)$ such that $f(x) = y$, and hence f is onto. □