1. Prove that for any natural numbers n and d there are unique q and r such that $n = q \cdot d + r$ where $-d \leq r < 0$.

   Proof. For the base step ($n = 0$) we write $0 = 1 \cdot d - d$ (here $r = -d$, which is allowd under this remainder scheme!). For the sake of induction now assume that for any $k < n$ we can write $k = q \cdot d + r$ with $-d \leq r < 0$. If $d \geq n$ we write $n = 1 \cdot d + (n - d)$, and note that $n \leq d$ implies $-d \leq n - d < 0$. If $d < n$ we write $n - d = q \cdot d + r$ with $-d \leq r < 0$ which we can do this by the induction hypothesis. Adding d to both sides we have $n = (q + 1) \cdot d + r$. ☐

2. Prove for $n \geq 4$ that $n^3 + 20 > n^2 + 15n$.

   Proof. For $n = 4$ we have $84 > 76$. For the sake of induction suppose that $n^3 + 20 > n^2 + 15n$ and consider

$$(n+1)^3 + 20 = n^3 + 3n^2 + 3n + 1 + 20$$
$$=(n^3 + 20) + (3n^2 + 3n + 1)$$

   and

$$\begin{aligned}(n+1)^2 + 15(n+1) &= n^2 + 2n + 1 + 15n + 15 \\ &= (n^2 + 15n) + (2n + 16).\end{aligned}$$

   By induction we have $n^3 + 20) > (n^2 + 15n)$ and $n \geq 4$ implies $3n^2 > n^2 \geq 16$ and $3n + 1 > 2n$. Thus putting the two larger together will remain larger, implying

$$(n+1)^3 + 20 > (n+1)^2 + 15(n+1)$$

   which establishes our result by induction that for $n \geq 4$:

$$n^3 + 20 > n^2 + 15n$$

   . ☐

3. Prove that $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$.

   Proof. For the base case let $n = 1$, the left hand side becomes $1(1 + 1) = 2$ and the right hand side is $\dfrac{1(2)(3)}{3} = 2$, and hence the result holds.

For the sake of induction suppose that the equality in question holds for n, and consider the sum $\sum_{i=1}^{n+1} i(i+1)$. Now $\displaystyle\sum_{i=1}^{n+1} i(i+1) = \left(\sum_{i=1}^{n} i(i+1)\right) + (n+1)(n+2)$. Using the induction hypothesis, by substitution we have

$$\begin{aligned}
\sum_{i=1}^{n+1} i(i+1) &= \left(\sum_{i=1}^{n} i(i+1)\right) + (n+1)(n+2) \\
&= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\
&= \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3} \\
&= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} \\
&= \frac{(n+1)(n+2)[n+3]}{3} = \frac{(n+1)(n+2)(n+3)}{3}
\end{aligned}$$

which is the desired result. Thus the result is true for all n by induction. □

4. For $n \in \mathbb{N}$ prove that if $x < y$ then $x^{2n-1} < y^{2n-1}$.

   It was intended here that $0 < x < y$ to avoid turning this into an exercise in handling special cases. We prove this result, which is an exercise in induction.

   Proof. For the base case we let $n = 1$ and have $2n - 1 = 1$, so $x^{2n-1} = x$ and $y^{2n-1} = y$, and the result holds by the hypothesis that $x < y$.

   Now assume that $x^{2n-1} < y^{2n-1}$ and consider $x^{2(n+1)-1} = x^{2n+1}$ and $y^{2(n+1)-1} = y^{2n+1}$. Note that if $x < y$, it follows that $x^2 < xy$ by multiplying both sides of the inequality by $x > 0$, and $xy < y^2$ by multiplying both sides of the inequality by $y > 0$, which together imply by the transitive property that $x^2 < y^2$.

   Now $x^{2n-1} < y^{2n-1}$ and so $x^{2n+1} = x^{2n-1}x^2 < y^{2n-1}x^2$. Further since $x^2 < y^2$ we know $y^{2n-1}x^2 < y^{2n-1}y^2 = y^{2n+1}$. Putting these together with the transitive property we have $x^{2n+1} < y^{2n+1}$ as desired, and the result thus holds by induction. □