Test One Review

1. Give a truth table for $[\sim (\sim p \vee q)] \wedge (p \vee r)$. Include enough intermediate columns so that I can tell that you know what you are doing.

p	q	r	$\sim p \vee q$	$p \vee r$	$[\sim (\sim p \vee q)] \wedge (p \vee r)$
T	T	T	T	T	F
T	T	F	T	T	F
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	F
F	T	F	T	F	F
F	F	T	T	T	F
F	F	F	T	F	F

2. Let $S = \{x \in \mathbb{R} | x^2 < x\}$ and let $T = \{x \in \mathbb{R} | 0 < x < 1\}$. Prove that $S = T$.

Proof. Let $x \in S$. Then $x^2 < x$. Note that $0^2 = 0$ and $1^2 = 1$, so we may conclude that $x \neq 0$ and $x \neq 1$. Three possibilities remain, $x < 0$, $0 < x < 1$, and $x > 1$. Now $x < 0$ would imply $x^2 > 0 > x$, contradicting $x^2 < x$, so we are forced to rule out the possiblility that $x < 0$. Consider again that if $x^2 < x$ then $x^2 - x < 0$, which gives us $x(x-1) < 0$ by factoring. If $x > 1$ then both x and $x - 1$ are positive, and hence their product would be positive, and hence $x(x-1) > 0$, contradicting $x(x-1) < 0$. Thus we must conclude that $x > 1$ is also impossible. The only possible case then is $0 < x < 1$, which implies that $x \in T$. Thus we've established that if $x \in S$ then $x \in T$, and hence $S \subseteq T$.

To show containment in the other direction, suppose that $y \in T$. Then $0 < y < 1$, and in particular, $y < 1$. Since $y > 0$, multiplying both sides of this inequality by y does not change its direction, so $y^2 < y$, and thus $y \in S$, and hence $T \subseteq S$.

Having established that $S \subseteq T$ and $T \subseteq S$ we now know that $S = T$. □

3. Prove that for any two sets A and B,

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

Proof. Let $x \in (A \setminus B) \cup (B \setminus A)$. Two cases arise. Consider first the possibility that $x \in (A \setminus B)$. Then $x \in A$ but $x \notin B$. The first of these implies that $x \in (A \cup B)$ while the second prevents x from being an element of $A \cap B$), so we have $x \in (A \cup B) \setminus (A \cap B)$. Now for the second case assume $x \in (B \setminus A)$. This means that $x \in B$ and $x \notin A$. The first of these again implies that $x \in (A \cup B)$ while the second of these again implies that $x \notin (A \cap B)$, and thus $x \in (A \cup B) \setminus (A \cap B)$. We have now established that in either case $x \in (A \setminus B) \cup (B \setminus A)$ implies $x \in (A \cup B) \setminus (A \cap B)$, so we have $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$.

Now to show containment in the other direction suppose that $y \in (A \cup B) \setminus (A \cap B)$. Then $y \in (A \cup B)$ and $y \notin (A \cap B)$. The second of these implies that y cannot be simultaneously an element of A and B, so y must be in A but not B, B but not A, or neither A nor B. With this in mind, the fact that $y \in (A \cup B)$ presents two cases, $y \in A$ or $y \in B$. Now if $y \in A$ it must be the case from the above discussion that $y \notin B$, hence $y \in (A \setminus B) \subseteq (A \setminus B) \cup (B \setminus A)$. In the second case, if $y \in B$ then the above discussion implies that $y \notin A$, hence $y \in (B \setminus A) \subseteq (A \setminus B) \cup (B \setminus A)$. We have in either case then that $y \in (A \setminus B) \cup (B \setminus A)$, and have thus established that $(A \cup B) \setminus (A \cap B) \subseteq (A \cup B) \setminus (A \cap B)$. □

4. Negate the following statements:

   (a) Every prime number is odd.
   Some prime numbers are not odd. (Some prime numbers are even.)

   (b) There is a natural number n such that $n(n+1) = n$ or $n(n+1) = n+1$.
   For all natural numbers n, $n(n+1) \neq n$ and $n(n+1) \neq n+1$.

   (c) $|x| = x$ or $|x| = -x$.
   $|x| \neq x$ and $|x| \neq -x$.

   (d) If n is even and if $5|n$ then the last digit of n is zero.
   n is even and $5|n$ but the last digit of n is non-zero.

5. Prove that if x is even and y is even then xy is divisible by 4.

   Proof. Suppose that x and y are even. Then there are integers m and n such that $x = 2m$ and $y = 2n$. Consider, then, xy. By substitution $xy = 2m2n = 4(mn)$, and thus is divisible by 4. □

6. Let a, b, and c be positive integers. Prove by contradiction that if $a \nmid b$ then $ac \nmid bc$.

   Proof. Suppose that $x \nmid b$ but $ac|bc$. Then there is an integer q such that $acq = bc$. Then $acq - bc = 0$, or $acq - bc = c(aq - b) = 0$. Since $c \neq 0$ it must be the case that $aq - b = 0$, so $aq = b$ and $a|b$, contradicting the assumption that $a \nmid b$, and we have established our result by contradiction. □

7. For natural numbers m and n, prove that if $mn \neq 1$ then $m \neq 1$ or $n \neq 1$. (Prove this by contrapositive). [This is how it should have read – sorry for the typo.]

   Proof. By contrapositive. Suppose that $m = 1$ and $n = 1$. Then $mn = 1$.

8. Consider the statement "If $a < 0$ and $b > 0$ then $ab < 0$."

   (a) Write the converse of the statement.
   If $ab < 0$ then $a < 0$ and $b > 0$.

(b) Write the contrapositive of the statement.
If $ab \geq 0$ then $a \geq 0$ or $b \leq 0$.

(c) Write the inverse of the statement.
If $a \geq 0$ or $b \leq 0$ then $ab \geq 0$.

(d) Write the negation of the statement.
$a < 0$ and $b > 0$ and $ab \geq 0$.