1. Let R is a relation on a set A such that for all $a \in A$, aRx for some $x \in A$ (i.e. everybody is related to somebody). Prove that if R is symmetric and transitive, then R is reflexive.

   Let $a \in A$. Since aRx for some $x \in A$ we know by symmetry that xRa as well. Applying the transitive property we have aRx and xRa implying aRa, and hence R is transitive.

2. Consider the relation R on the real numbers $\mathbb{R}$ given by aRb when $|a - b| < \epsilon$ for some fixed positive number ϵ. Is R reflexive? Is R symmetric? Is R transitive?

   R is reflexive since $|a - a| = 0 < \epsilon$. R is symmetric since if aRb, $|a - b| < \epsilon$, and hence $|b - a| = |a - b| < \epsilon$, so bRa. To see that R is not transitive, note that $0R(\frac{2}{3}\epsilon)$ and $(\frac{2}{3}\epsilon)R(\frac{4}{3}\epsilon)$, but $|\frac{4}{3}\epsilon - 0| = \frac{4}{3}\epsilon > \epsilon$.

3. Consider the set S of all one-to-one and onto functions from the set $\{1, 2, 3\}$ to the set $\{4, 5, 6\}$. For example f $(f(1) = 6, f(2) = 5, f(3) = 4)$ and g $(g(1) = 5, g(2) = 4, g(3) = 6)$. Write down all such functions and prove that your list is exhaustive (i.e. that you have indeed written down all of them).

x	1	2	3
$f(x)$	4	5	6

x	1	2	3
$f(x)$	4	6	5

x	1	2	3
$f(x)$	5	4	6

x	1	2	3
$f(x)$	5	6	4

x	1	2	3
$f(x)$	6	4	5

x	1	2	3
$f(x)$	6	5	4

   To see that this list is exhaustive note that there are three choices for $f(1)$, either $f(1) = 4$, $f(1) = 5$, or $f(1) = 6$. If $f(1) = n$, it must be the case that neither $f(2)$ nor $f(3)$ will equal n, leaving only two choices for $f(2)$. Having fixed $f(1)$ and $f(2)$ there is only one option for $f(3)$, thus there are $3 \cdot 2 \cdot 1$ possible functions, and all six are listed above.

4. Consider the set S above. Note that for any $f \in S$, for $i < j$ either $f(i) < f(j)$ or $f(i) > f(j)$ since $f(i) = f(j)$ is impossible due to the one-to-one property. If $f(i) < f(j)$ we say that f preserves the orientation of i and j. If $f(i) > f(j)$ we say f reverses the orientation of i and j. Define $\sim$ on S as follows: $f \sim g$ if f reverses the same number of orientations (out of all possible pairs i, j as g). Prove that $\sim$ is indeed an equivalence relation and write down the equivalence classes.

Proof. Of course $\sim$ is reflexive since any f reverses the same number of orientations as itself. Similarly if f reverses the same number of orientations as g then g reverses the same number of orientations as f, so $f \sim g$ implies $g \sim f$ and $\sim$ is symmetric. To see that f is transitive suppose that $f \sim g$ and $g \sim h$. Then f reverses the same number of orientations as g, let's call this number k. Now since $g \sim h$ it must be the case that h also reverses k orientations, which means that, since both f and h reverse k orientations, $f \sim h$ and $\sim$ is transitive. □

The equivalence classes are:

x	1	2	3
$f(x)$	4	5	6

Reverses 0

x	1	2	3
$f(x)$	4	6	5

x	1	2	3
$f(x)$	5	4	6

Reverse 1

x	1	2	3
$f(x)$	5	6	4

x	1	2	3
$f(x)$	6	4	5

Reverse 2

x	1	2	3
$f(x)$	6	5	4

Reverses 3