1. Let $f : A \to A$ be a function. Prove that if $f \circ f$ is injective then $f$ is injective.

   *Proof.* Suppose that $f(a) = f(b)$ for some $a, b \in A$. To show that $f$ is injective we need to show that $a = b$. Now if $f(a) = f(b)$, we have $f(f(a)) = f(f(b))$ (we're plugging in the same thing, and $f$ is a function). This gives us that $f \circ f(a) = f \circ f(b)$, and since $f \circ f$ is one to one, $a = b$. □

2. Consider $f : A \to B$ and $g : B \to A$. Prove or give a counterexample: If $f(g(b)) = b$ for all $b \in B$ then $f$ is one-to-one and onto.

   It is clear that $f$ must be onto, since for any $b \in B$, $g(b) \in A$ and $f(g(b)) = b$. However for a counterexample to one-to-one consider the following:

   $$A = \{a_1, a_2\}, \quad B = \{b\}, \qquad g(b) = a_1, \quad f(a_1) = f(a_2) = b.$$

   One sees that $f(g(b)) = b$ for all (the only!) $b \in B$, but $f$ is not one-to-one. □

3. Consider $f : A \to B$ and $g : B \to A$. Prove or give a counterexample: If $g(f(a)) = a$ for all $a \in A$, then $f(g(b)) = b$ for all $b \in B$.

   We modify slightly the above counterexample:

   $$A = \{a\}, \quad B = \{b_1, b_2\}, \qquad f(a) = b_1, \quad g(b_1) = g(b_2) = a.$$

   We have $g(f(a)) = g(b_1) = a$ but $f(g(b_2)) = f(a) = b_1 \neq b_2$. □

4. Let $\mathbb{N}^+ = \{1, 2, 3, 4, \ldots\}$ (some authors think of this as $\mathbb{N}$, while others think of this as $\mathbb{N} \setminus \{0\}$). Which of the following define onto functions from $\mathbb{N}^+ \times \mathbb{N}^+$ to $\mathbb{N}^+$?

   (a) $f((a, b)) = a + b$
   Note that since both $a > 0$ and $b > 0$ we must have $f(a, b) = a + b > a$ and $f(a, b) = a + b > b$. Then since $a, b \in \mathbb{N}^+$ implies that $a \geq 1$ and $b \geq 1$ we have $f(a, b) > 1$, implying $f(a, b) \neq 1$ for any choice of $a, b$, and thus $f$ is not onto. □

   (b) $g((a, b)) = ab$
   Let $n \in \mathbb{N}^+$. Then $f(1, n) = n$, and $f$ is onto. □

   (c) $h((a, b)) = ab(b + 1)/2$
   Let $n \in \mathbb{N}^+$. Then $f(n, 1) = n(1)(1 + 1)/2 = n$, and $f$ is onto. □

5. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that there is a positive real numbers $c$ and $n$ such that $|f(x) - f(y)| \geq c|x - y|^n$ for all $x, y \in \mathbb{R}$. Prove that $f$ is one to one.

   *Proof.* Suppose that $f(x) = f(y)$. To show that $f$ is one-to-one we must show that $x = y$. Now $|f(x) - f(y)| = 0 \geq c|x - y|^n$. Now $c > 0$, and $|x - y| \geq 0$, so $|x - y|^n \geq 0$, and hence we must conclude that $c|x - y|^n = 0$ since it can't be negative. Of course $c > 0$, so we must have $|x - y|^n = 0$, which implies, since a positive number to any power remains positive, that $|x - y| = 0$, and hence $x = y$ and $f$ is one-to-one. □