

Prove that $(A \heartsuit B) \heartsuit C = A \heartsuit (B \heartsuit C)$

$$\begin{aligned}
 \textcircled{6}. (A \heartsuit B) \heartsuit C &= ((A-B) \cup (B-A) - C) \cup (C - ((A-B) \cup (B-A))) \\
 &= (((A \cap \bar{B}) \cup (B \cap \bar{A})) \cap \bar{C}) \cup (C \cap ((A \cap \bar{B}) \cup (B \cap \bar{A}))) \\
 &= (((A \cap \bar{B}) \cup (B \cap \bar{A})) \cap \bar{C}) \cup (C \cap ((A \cap \bar{B}) \cap (B \cap \bar{A}))) \\
 &= (((A \cap \bar{B}) \cup (B \cap \bar{A})) \cap \bar{C}) \cup (C \cap ((\bar{A} \cup B) \cap (\bar{B} \cap A))) \\
 &= (((A \cap \bar{B}) \cup (B \cap \bar{A})) \cap \bar{C}) \cup (C \cap ((\bar{A} \cap \bar{B}) \cup (\bar{A} \cap A) \cup (B \cap \bar{B}) \cup (B \cap A))) \\
 &= (((A \cap \bar{B} \cap \bar{C}) \cup (B \cap \bar{A} \cap \bar{C})) \cup (C \cap ((\bar{A} \cap \bar{B}) \cup (B \cap A)))) \\
 &= ((A \cap \bar{B} \cap \bar{C}) \cup (B \cap \bar{A} \cap \bar{C})) \cup ((C \cap \bar{A} \cap \bar{B}) \cup (C \cap B \cap A)) \\
 &= (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C) \cup (A \cap B \cap C)
 \end{aligned}$$

$$\begin{aligned}
 A \heartsuit (B \heartsuit C) &= (A - ((B-C) \cup (C-B))) \cup (((B-C) \cup (C-B)) - A) \\
 &= (A \cap ((B \cap \bar{C}) \cup (C \cap \bar{B}))) \cup (((B \cap \bar{C}) \cup (C \cap \bar{B})) \cap \bar{A}) \\
 &= (A \cap ((\bar{B} \cap \bar{C}) \cap (C \cap \bar{B}))) \cup (((B \cap \bar{C}) \cup (C \cap \bar{B})) \cap \bar{A}) \\
 &= (A \cap ((\bar{B} \cup C) \cap (\bar{C} \cup B))) \cup (((B \cap \bar{C}) \cup (C \cap \bar{B})) \cap \bar{A}) \\
 &= (A \cap ((\bar{B} \cap \bar{C}) \cup (B \cap \bar{B}) \cup (C \cap \bar{C}) \cup (C \cap B))) \cup (((B \cap \bar{C}) \cup (C \cap \bar{B})) \cap \bar{A}) \\
 &= (A \cap ((\bar{B} \cap \bar{C}) \cup (C \cap B))) \cup (((B \cap \bar{C}) \cup (C \cap \bar{B})) \cap \bar{A}) \\
 &= (A \cap \bar{B} \cap \bar{C}) \cup (A \cap B \cap C) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)
 \end{aligned}$$

$$\begin{aligned}
 \text{Since } &(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C) \cup (A \cap B \cap C) \\
 &= (A \cap \bar{B} \cap \bar{C}) \cup (A \cap B \cap C) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)
 \end{aligned}$$

we have shown that $(A \heartsuit B) \heartsuit C = A \heartsuit (B \heartsuit C)$ and therefore the operation \heartsuit on $P(A)$ is associative. \square