1. Let $S = \{0123456\}$, and consider the operation $\cdot$ on S given by $x \cdot y = xy \mod 7$. Write a Cayley table for this operation. Is there an identity element? If so, does every element have an inverse?

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

We see that 1 is an identity element since $1 \cdot n = n \cdot 1 = n$ for all $n \in \{0, 1, 2, 3, 4, 5, 6\}$. We also note that 1 is its own inverse, as is 6, since $1 \cdot 1 = 6 \cdot 6 = 1$. Furthermore 2 and 4 are inverses since $2 \cdot 4 = 4 \cdot 2 = 1$ and 3 and 5 are inverses since $3 \cdot 5 = 5 \cdot 3 = 1$. 0 has no inverse since there is no x with $0 \cdot x = 1$. □

2. Let A be a set and consider the operation $\heartsuit$ on $\mathcal{P}(A)$ given by $S \heartsuit T = (S \setminus T) \cup (T \setminus S)$.

   (a) Prove that $\heartsuit$ is a binary operation on $\mathcal{P}(A)$.

   Proof. Given $S, T \in \mathcal{P}(A)$ we note that $S \setminus T \subseteq S \subseteq A$ and $T \setminus S \subseteq T \subseteq A$, which implies that $(S \setminus T) \cup (T \setminus S) \subseteq A$, and hence is an element of $\mathcal{P}(A)$. It is thus clear then, that $\heartsuit$ is a function $\heartsuit : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathcal{P}(A)$, and thus is a binary operation. □

   (b) Is $\heartsuit$ associative? (prove or give a counterexample)
   I will get this one typeset soon.

   (c) Is $\heartsuit$ commutative? (prove or give a counterexample)

   Proof.
   $$R \heartsuit S = (R \setminus S) \cup (S \setminus R) = (S \setminus R) \cup (R \setminus S) = S \heartsuit R$$
   □

   (d) Is there an identity element for $\heartsuit$ (if so – prove that there is, if not give a counterexample of a set A for which $\heartsuit$ has no identity element)? If there is always an identity element for $\heartsuit$, for any $B \subseteq A$ is there an inverse B^{-1} such that $B \heartsuit B^{-1}$ equals this identity element? (prove or give a counterexample)
   We claim that $\emptyset$ is an indentity element for $\heartsuit$ and that $S^{-1} = S$ for all $S \in \mathcal{P}(A)$.

   Proof. First note that $S \setminus \emptyset = S$ and that $\emptyset \setminus S = \emptyset$. Thus $S \heartsuit \emptyset = S \cup \emptyset = S$, and thus $\emptyset$ is an identity element for the operation $\heartsuit$.
   Now consider $S \heartsuit S$. Since $S \setminus S = \emptyset$ we have that $S \heartsuit S = \emptyset \cup \emptyset = \emptyset$, and hence every set is its own inverse under the operation $\heartsuit$. □

3. Prove that there are no onto functions from [4] to [5]. (Recall that $[n] = \{1, 2, 3, \ldots, n\}$.)

Proof. Suppose that $f : [4] \to [5]$ is onto. Then for each $i \in \{1, 2, 3, 4, 5\}$ there is some x_i in [4] such that $f(x_i) = i$. Each of $[1, 2, 3, 4]$ can appear but once in the list $[x_1, x_2, x_3, x_4, x_5]$ since f is a function, but since the length of $[x_1, x_2, x_3, x_4, x_5]$ is five and there are only four elements in $[1, 2, 3, 4]$ one of these elements must be repeated, contradicting the fact that there can be no repeats. Thus $f : [4] \to [5]$ cannot be onto. □

4. Consider the set $\{0, 1, 2, 3, 4\}$ and the operation $*$ on it given by $x * y = 2y - x \mod 5$. Write a Cayley table for this operation. Is it commutative? Is is associative?

The Cayley table follows.

$*$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

Now $0 * 1 = 2$ and $1 * 0 = 4$, so $*$ is not commutative. Furthermore $(0 * 1) * 2 = 2 * 2 = 2$ but $0 * (1 * 2) = 0 * 3 = 1$, so $*$ is not associative either. □