Foundations Test Two

1. Let $f : A \to B$ be a function and let $S, T \subseteq A$. Prove that $f(S \cap T) \subseteq f(S) \cap f(T)$.

Proof. Let $b \in f(S \cap T)$. Then there is $a \in S \cap T$ with $f(a) = b$. Now since $a \in S$, $b = f(a) \in f(S)$. Also since $a \in T$, $b = f(a) \in f(T)$. Thus $f(a) \in f(S) \cap f(T)$. $\square$

2. Let $S = \{a, b, c\}$. Write down all elements of $\mathcal{P}(X)$?

- $\emptyset$
- $\{a\}$
- $\{b\}$
- $\{c\}$
- $\{a, b\}$
- $\{a, c\}$
- $\{b, c\}$
- S

3. Consider the Cayley table for the operation $*$ below:

$*$	a	b	c	d	e
a	b	e	d	a	c
b	d	e	d	b	c
c	e	d	b	c	a
d	a	b	c	d	e
e	c	a	b	e	d

(a) Is $*$ commutative? (why or why not?)

No, $a * b = e$ and $b * a = d$. If $*$ was commutative these would be equal.

(b) Is $*$ associative? (why or why not?)

No, $(a * b) * c = e * c = b$ but $a * (b * c) = a * d = a$. If $*$ was commutative these would be equal.

(c) Is there an identity element for $*$? (why or why not)?

Yes, d is an identity element because $x*d = x$ and $d*x = x$ for all $x \in \{a, b, c, d, e\}$.

(d) Which, if any, elements of the set $\{a, b, c, d, e\}$ have inverses?

a has no inverse since $a * c = d$ but $c * a \neq d$ and $b * a = d$ but $a * b = e$.
$b * c = c * b = d$ so b and c are inverses.
$d * d = d$ so d is its own inverse.
$e * e = d$ so e is its own inverse.

4. Let $\sigma = (1\ 3\ 4\ 5)$ and $\tau = (2\ 5\ 3)$ be elements of S_5. Compute $\sigma\tau$ and $\tau\sigma$ (write your answers in cycle notation).

$\sigma\tau = (1\ 3\ 4\ 5)\ (2\ 5\ 3) = (1\ 3\ 2)\ (4\ 5)$

$\tau\sigma = (2\ 5\ 3)\ (1\ 3\ 4\ 5) = (1\ 2\ 5)\ (3\ 4)$

5. Consider the following function f from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$:

x	1	2	3	4	5
$f(x)$	4	5	1	3	2

Write f in cycle notation.

$f = (1\ 4\ 3)\ (2\ 5)$

6. Let $\diamondsuit$ be a binary operation on a set S. Prove that there can be only one identity element of $\diamondsuit$. (Suppose that s and e are both identity elements. Prove that $s = e$.)

Proof. Suppose that s and e are both identity elements. Since s is an identity element, $s \diamondsuit e = e$. But e is an identity element so $s \diamondsuit e = s$. Thus we have $e = s \diamondsuit s = s$. $\square$

7. Let $f : A \to C$ and $g : B \to D$ be functions. Define $h : A \times B \to C \times D$ by $h((a,b)) = (f(a), g(b))$. Prove that if f and g are onto then h is onto as well.

Proof. Let $(c,d) \in C \times D$. Since f is onto there is $a \in A$ with $f(a) = c$, and since g is onto there is $b \in B$ with $g(b) = d$. Then $h((a,b)) = ((f(a), g(b)) = (c,d)$, and hence h is onto. $\square$

8. Define the relation $\sim$ on $\mathbb{N}_{\geq 2}$ as follows: $m \sim n$ if whenever any number k divides m then it is the case that k also divides n. Prove that $\sim$ is a partial order relation on $\mathbb{N}_{\geq 2}$. (By $\mathbb{N}_{\geq 2}$ we mean $\{n \in \mathbb{N} | n \geq 2\}$. You are free to use, without proof, the basic facts about divisibility that you know from grammar school, even though we've only proved some of these in class.)

Proof. Let $n \in \mathbb{N}_{\geq 2}$. Clearly any k that divides n divides n, so $n \sim n$, and $\sim$ is reflexive.

Now let $m, n \in \mathbb{N}_{\geq 2}$ with $m \sim n$ and $n \sim m$. Then since $m \sim n$ and $m|m$ it must be the case that $m|n$. But since $n \sim m$ and $n|n$ it must be the cast that $n|m$. At this point we now have $m|n$ and $n|m$, implying that $m = n$ and $\sim$ is antisymmetric.

Finally let $\ell, m, n \in \mathbb{N}_{\geq 2}$ with $\ell \sim m$ and $m \sim n$. Let k divide ℓ. To see that $\sim$ is transitive we must show that k divides n. Now since $\ell \sim m$ and $k|\ell$ we have that $k|m$. But now since k is a factor of m and $m \sim n$ it follows that $k|n$ as desired, and hence $\sim$ is transitive. □

9. Let $f : A \to B$ be a function and let $S \subset A$. Prove that $f^{\hookleftarrow}(\overline{f(S)}) \subseteq \overline{S}$. Give a counterexample to show that the reverse containment need not be true.

Proof. Let $a \in f^{\hookleftarrow}(\overline{f(S)})$. Then $f(a) \in \overline{f(S)}$. Now if $a \in S$ we would have $f(a) \in f(S)$, contradicting the fact that $f(a) \in \overline{f(S)}$, and so it follows that $a \in \overline{S}$. □

For a counterexample to the reverse containment consider the following situation:

$$A = \{1, 2, 3\}, \quad S = \{1\}, \quad B = \{a, b\}, \qquad f(1) = a, f(2) = b, f(3) = b.$$

Now $f(S) = \{b\}$, so $\overline{f(S)} = \{a\}$. Then $f^{\hookleftarrow}(\overline{f(S)}) = f^{\hookleftarrow}(\{a\}) = \{1\}$, but $\overline{S} = \{1, 2\}$, which is not a subset of $\{1\}$. □