1. Let V be the collection of polynomials with coefficients in $\mathbb{Q}$ in the variable x of degree at most 5. Prove that V is a vector space over $\mathbb{Q}$ of dimension 6 with basis $\{1, x, x^2, x^3, x^4, x^5\}$. Prove that $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3, 1+x+x^2+x^3+x^4, 1+x+x^2+x^3+x^4+x^5\}$ is a basis for V as well.

2. In class we proved the theorem: $dim(V) = dim(W)+dim(V/W)$ where W is a subspace of V, and deduced the corollary: if $f : V \to W$ then $dim(V) = dim(ker(f)) + dim(Im(f))$. Prove this in reverse, that is, prove (without appeal to the theorem above) theorem: that if $f : V \to W$ then $dim(V) = dim(ker(f)) + dim(Im(f))$ and deduce corollary: if $W \leq V$ then $dim(V) = dim(W) + dim(V/W)$.

3. Let V be a vector space over a field k and let $f : V \to V$ be a linear transformation. A non-zero vector $v \in V$ such that $f(v) = \lambda v$ for some constant $\lambda \in k$ is called an *eigenvector* with *eigenvalue* λ. Prove that for a fixed $\lambda \in k$ the collection of eigenvectors of f with eigenvalue λ (together with $\vec{0}$) forms a subspace of V.

4. Let ϕ be a linear transformation from a finite dimensional vector space V to itself (i.e. $\phi : V \to V$). Prove that there is an integer m such that $\ker(\phi^m) \cap \mathrm{Im}(\phi^m) = \{0\}$.

5. Let V be the vector space of polynomials with coefficients in $\mathbb{Q}$ in the variable x of degree at most 5. Determine the transition matrix from the basis $\{1, x, x^2, x^3, x^4, x^5\}$ to the basis $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3, 1+x+x^2+x^3+x^4, 1+x+x^2+x^3+x^4+x^5\}$.

6. Let V be the vector space of polynomials with coefficients in $\mathbb{Q}$ in the variable x of degree at most 5. Let $d : V \to V$ be the linear transformation of V to itself given by the usual differentiation of a polynomial with respect to x (i.e. $d(p(x)) = p'(x)$. Determine the matrix of d with respect to the two bases for V given in the previous problem.

7. Let R be commutative and let ${}_RM$ be a cyclic left R-module generated by m. Prove that $Ann_R(M) = Ann_R(m)$. Conclude that $M \cong R/Ann_R(M)$.

8. Prove that if R is commutative and $M = Rx$ and $N = Ry$ are cyclic left R-modules then $M \cong N$ if and only if $Ann_R(M) = Ann_R(N)$.

9. Let ${}_RM$ be free with basis $\{m_i\}$ and let ${}_RN$ be any R-module. Prove that for any set function $f : \{m_i\} \to N$ there is a unique R-homomorphism $f' : M \to N$ such that $f'(m_i) = f(m_i)$ for all i.

10. Let A be the matrix

$$\begin{pmatrix} -2 & 3 & 0 \\ -3 & 3 & 0 \\ -12 & 12 & 6 \end{pmatrix}$$

over the integers. Find the Smith normal form for A (hint – start by adding column 2 to column 1).

11. Continuing the above problem, find the matrices P and Q such that QAP is the Smith normal form.

12. Still continuing, if the original basis for G was $\{g_1, g_2, g_3\}$ and the original set of generators for K was $\{k_1, k_2, k_3\}$, find the new basis and set of generators.